What is the improper name to call you: Key

AP Calculus BC: 10.9 Improper Integrals with discontinuities

Evaluate each integral below

1. $\int_0^1 \frac{3}{\sqrt[3]{x^2}}\,dx$ discontinuous @ $x=0$

$\lim_{a\to 0^+}\int_a^1 3x^{-2/3}\,dx = \lim_{a\to 0^+}\left[\frac{(3)3x^{1/3}}{1}\right]_a^1$

$\lim_{a\to 0^+}\left[9\sqrt[3]{a}\right]_a^1 = 9\sqrt[3]{1} - \lim_{a\to 0^+} 9\sqrt[3]{x}$

$9 - 9\sqrt[3]{0}$

$\boxed{9}$

2. $\int_0^4 \frac{1}{x}\,dx = \lim_{b\to 0^+}\left[\ln|x|\right]_b^4$ discontinuous @ $x=0$

$\ln|4| - \lim_{b\to 0^+}\ln|b|$

$\ln|4| - (-\infty) = \boxed{\infty}$

$\boxed{\text{divergent}}$

discontinuous @ $x=3$

3. $\int_3^4 \frac{1}{\sqrt{x-3}}\,dx = \lim_{a\to 3^+}\int_a^4 (x-3)^{-1/2}\,dx$

$\lim_{a\to 3^+}\left[\frac{2(x-3)^{1/2}}{1}\right]_a^4 = 2(4-3)^{1/2} - \lim_{a\to 3^+} 2(a-3)^{1/2}$

$2 - 2(3-3)^{1/2}$

$\boxed{2}$

4. $\int_0^{\frac{1}{2}} \frac{2}{\sqrt{1-4x^2}}\,dx$ discontinuous @ $x = 1/2$

$\left[\sin^{-1}(2x)\right]_0^{1/2} = \lim_{b\to 1/2^-}\left[\sin^{-1}(2x)\right]_0^b$

$\lim_{b\to 1/2^-}\sin^{-1}(2(b)) - \cancel{\sin^{-1}(0)}$

$\sin^{-1}(1)$

$\boxed{\pi/2}$

discontinuous @ $x=0$

5. $\int_0^{\frac{\pi}{2}} \cot(x)\,dx = \int_0^{\pi/2}\frac{\cos(x)}{\sin(x)}\,dx$

$\int_0^{\pi/2}\frac{\cos(x)}{u}\frac{du}{\cos(x)} = \int_0^{\pi/2}\frac{1}{u}\,du$ $\quad u=\sin(x)$, $du=\cos(x)\,dx$, $dx=\frac{du}{\cos(x)}$

$\left[\ln|u|\right]_0^{\pi/2} = \lim_{a\to 0^+}\left[\ln|\sin(x)|\right]_a^{\pi/2}$

$\underbrace{\ln|\sin(\pi/2)|}_{0} - \lim_{a\to 0^+}\ln|\sin(a)| = --\infty = \boxed{\infty}$

$\boxed{\text{divergent}}$

6. $\int_{-2}^2 \frac{1}{\sqrt{4-x^2}}\,dx = \int_{-2}^2 \frac{1}{\sqrt{4(1-\frac{1}{4}x^2)}}\,dx = \int_{-2}^2 \frac{1}{2\sqrt{1-(\frac{1}{2}x)^2}}\,dx$

$\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^2$ discontinuous @ $x=-2$ & 2

$\lim_{a\to -2^+}\lim_{b\to 2^-}\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_a^b$

$\lim_{b\to 2^-}\sin^{-1}\left(\frac{b}{2}\right) - \lim_{a\to -2^+}\sin^{-1}\left(\frac{x}{2}\right)$

$\frac{\pi}{2} - -\frac{\pi}{2} = \boxed{\pi}$

7. Find the area under the curve of the function $f(x) = \frac{1}{(x-4)^3}$ for the interval [3,6].

discontinuous @ $x=4$

$\int_3^4 \frac{1}{(x-4)^3}\,dx + \int_4^6 \frac{1}{(x-4)^3}\,dx = \left[\frac{-1}{2(x-4)^2}\right]_3^4 + \left[\frac{-1}{2(x-4)^2}\right]_4^6 =$

$\left[-\infty - -\frac{1}{2}\right] + \left[-\frac{1}{8} - \infty\right] = \boxed{-\infty}$

$\boxed{\text{divergent}}$

$\int \frac{1}{(x-4)^3}\,dx = \int (x-4)^{-3}\,dx$ $\quad u=x-4$, $du=dx$, $dx=du$

$\left[\frac{(x-4)^{-2}}{-2}+C\right] = \left[\frac{-1}{2(x-4)^2}+C\right]$